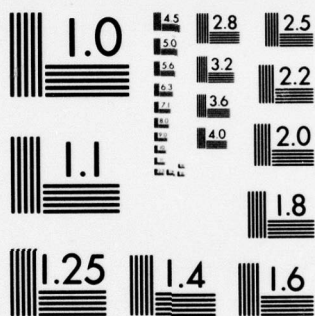| OF |
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⑥ ON THE TRANSVERSE TWISTING OF SHALLOW
SPHERICAL RING CAPS

⑩ by E./Reissner

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ON THE TRANSVERSE TWISTING OF SHALLOW

SPHERICAL RING CAPS

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ABSTRACT

The problem of transverse twisting of a shallow spherical
shell with a small circular hole is solved, in generalization
of the corresponding problem of a flat plate. The solution is
of interest as a closed-form solution of an unsymmetrical stress
concentration problem, with quantitative features depending on
its boundary layer behavior for large values of a relevant
parameter. The problem is also of interest as an example of
applicability of a previously proposed asymptotic procedure where
interior contributions and edge-zone contributions are determined
in sequence rather than simultaneously.

On the Transverse Twisting of Shallow Spherical Ring Caps[†]

by

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Introduction. The original aim of this note was to formulate
a nonrotationally—symmetric stress concentration problem for thin
shells which could be solved in closed form, and to obtain the solution
of this problem. It appeared in the course of the analysis that this
stress concentration problem was also a particularly fitting example
for the application of an asymptotic solution method for unsymmetric
shell problems, involving the concepts of interior and edge zone solu-
tion contributions and of the concept of contracted boundary conditions
for the separate determination of these contributions, which had been
proposed sometime earlier [4].

The problem is as follows. We consider an isotropic shallow spherical
shell with the edges defined by two pairs of mutually perpendicular planes
perpendicular to a base plane, with the corners of the rectangle in the
base plane which is determined by the two pairs of mutually perpendicular
planes coinciding with the corners of the shell boundary curve. Given
this configuration, we assume that the edges of the shell are free of
stress, except for the action of equal and opposite concentrated corner
forces, as indicated in Figure 1. Our object is the state of stress in
the shell, without or with a _small_ concentric circular hole at the apex.

It is evident that a limiting case of the above problem is the
corresponding problem of a flat plate, with the solution of the problem

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without the circular hole being a special case of the problem of
St. Venant torsion of narrow rectangular cross section beams, and with
the solution of the circular-hole problem being included in solutions
by Goodier for a class of transverse plate flexure problems [2].

In the present analysis the plate flexure problem appears upon
assuming the value of a certain parameter μ to be zero. At the same
time the asymptotic analysis corresponding to the procedure described
in [4] is appropriate for values of μ which are large compared to unity.
In the interim region of finite values of μ it is necessary to obtain
appropriate solutions of the equations of shell theory, which in this
instance may be taken from shallow-shell theory.

Regarding the physical aspects of the problem we find, as expected,
a dominance of bending stresses over membrane stresses in the interior
of the shell region. On the other hand, we also find that for suffi-
ciently large values of μ we have membrane stresses in an edge zone
which are of the same order of magnitude as the bending stresses in this
zone, in such a way that the value of the stress concentration factor
for this problem of transverse bending involves both bending and mem-
brane stresses in a significant manner.

Equations for Isotropic Homogeneous Shallow Spherical Shells.
We consider a shallow spherical shell with middle surface equation
$z = H - r^2/2R$, where R is the radius of the shell, H the distance of the
apex from the base plane of the shell, and r and θ are polar coordinates
in the base plane. We assume that the shell is free of distributed
surface forces and have then that tangential stress resultants N, stress
couples M and transverse stress resultants Q are expressed as follows
in terms of a stress function K and a transverse displacement function
w, [3],

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$$N_{rr} = \frac{K_{,r}}{r} + \frac{K_{,\theta\theta}}{r^2} \ , \quad N_{\theta\theta} = \nabla^2 K - N_{rr} \ , \quad N_{r\theta} = -\frac{K_{,\theta r}}{r} + \frac{K_{,\theta}}{r^2} \ , \tag{1}$$

$$M_{rr} = -D[\nabla^2 w - (1-\nu)(\frac{w_{,r}}{r} + \frac{w_{,\theta\theta}}{r^2})] \ , \quad M_{\theta\theta} = -(1+\nu)D\nabla^2 w - M_{rr} \ , \tag{2}$$

$$M_{r\theta} = -(1-\nu)D(\frac{w_{,\theta r}}{r} - \frac{w_{,\theta}}{r^2}) \ , \quad Q_r = -D(\nabla^2 w)_{,r} \ , \quad Q_\theta = -D\frac{(\nabla^2 w)_{,\theta}}{r} \ . \tag{3}$$

Use of appropriate equations of equilibrium and compatibility in conjunction with the above, and in conjunction with stress strain relations of the form $\varepsilon_{rr} = B(N_{rr} - \nu N_{\theta\theta})$, etc. leads to differential equations for K and w of the form

$$RB\nabla^2\nabla^2 K - \nabla^2 w = 0 \ , \quad RD\nabla^2\nabla^2 w + \nabla^2 K = 0 \ , \tag{4}$$

where $\nabla^2 = (\ )_{,rr} + r^{-1}(\ )_{,r} + r^{-2}(\ )_{,\theta\theta}$.

It is readily verified that the solution of the system (4) may be expressed in terms of three functions ϕ, ψ and χ in the form [4],

$$w = \phi + \chi \ , \quad K = \psi - RD\nabla^2\chi \ , \tag{5}$$

provided that

$$\nabla^2\phi = 0 \ , \quad \nabla^2\psi = 0 \ , \quad \nabla^2\nabla^2\chi + \lambda^4\chi = 0 \ , \tag{6}$$

where $\lambda^4 = 1/R^2 BD$.

We note for what follows as expressions for resultants and couples in terms of ϕ, ψ and χ

$$N_{rr} = -\psi_{,rr} - RD\left(\frac{(\nabla^2\chi)_{,r}}{r} + \frac{(\nabla^2\chi)_{,\theta\theta}}{r^2}\right) \ , \tag{7}$$

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$$N_{\theta\theta} = \psi_{,rr} + RD\left(\frac{(\nabla^2\chi)_{,r}}{r} + \frac{(\nabla^2\chi)_{,\theta\theta}}{r^2} + \lambda^4\chi\right) \quad , \tag{8}$$

$$N_{r\theta} = \frac{\psi_{,r\theta}}{r} - \frac{\psi_{,\theta}}{r^2} - RD\left(\frac{(\nabla^2\chi)_{,r\theta}}{r} - \frac{(\nabla^2\chi)_{,\theta}}{r^2}\right) \quad , \tag{9}$$

$$Q_r = -D(\nabla^2\chi)_{,r} \quad , \tag{10}$$

$$M_{rr} = -(1-\nu)D\phi_{,rr} - D\nabla^2\chi + (1-\nu)D\left(\frac{\chi_{,r}}{r} + \frac{\chi_{,\theta\theta}}{r^2}\right) \quad , \tag{11}$$

$$M_{\theta\theta} = (1-\nu)D\phi_{,rr} - \nu D\nabla^2\chi - (1-\nu)D\left(\frac{\chi_{,r}}{r} + \frac{\chi_{,\theta\theta}}{r^2}\right) \quad , \tag{12}$$

$$M_{r\theta} = (1-\nu)D\left(\frac{\phi_{,\theta}}{r^2} - \frac{\phi_{,r\theta}}{r}\right) + (1-\nu)D\left(\frac{\chi_{,\theta}}{r^2} - \frac{\chi_{,r\theta}}{r}\right) \quad , \tag{13}$$

and we also note the designations of ϕ and ψ as __inextensional bending__ and __membrane__ (interior) __solution contributions__, respectively, and the designation of χ as __edge zone solution contribution__, with the physical significance of the latter designation depending on an appropriate relation between the length-parameter $1/\lambda$ and an appropriate linear dimension of the shell.

The Boundary Value Problem. We start out with the observation that the classical solution $w = -Pxy/2(1-\nu)D$ for St. Venant twisting of a flat rectangular plate as produced by an arrangement of concentrated corner forces P, in conjunction with an assumption of no inplane stress, that is, in conjunction with the stipulation $K = 0$, also satisfies the differential equations (4) for shallow spherical shells. Furthermore, this solution of (4) satisfies the same corner force conditions for a spherical cap with otherwise free edges, in the event that the projection of these edges onto the base plane of the shell happens to be rectangular.

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Having the above simple solution for transverse twisting of a spherical cap, we ask for the way in which this solution is modified by the presence of a circular hole of radius a, concentric with the apex of the shell, given that a is small compared to the overall dimensions of the cap. Evidently, the boundary conditions for the free edge of this hole are of the from

$$r = a ; \quad N_{rr} = N_{r\theta} = M_{rr} = Q_r + r^{-1}M_{r\theta,\theta} = 0 . \tag{14}$$

As regards the boundary conditions along the outer edges of the cap, we make the stipulation that for large r we will have a homogeneous state of stress with cartesian couple and resultant components $M_{xy} = -P/2$, $M_{xx} = M_{yy} = 0$, $Q_x = Q_y = N_{xx} = N_{yy} = N_{xy} = 0$. This is transformed, in an elementary manner, into four conditions of the form

$$r \to \infty ; \quad M_{rr} = \frac{1}{2} P \sin 2\theta , \quad Q_r + r^{-1}M_{r\theta,\theta} = N_{rr} = N_{r\theta} = 0 . \tag{15}$$

Closed-Form Solution of the Boundary Value Problem. The form of the boundary conditions (14) and (15), in conjunction with the form of the differential equations (4) indicates that suitable expressions for w and K will be product solutions $f(r) \sin 2\theta$. Considering that w and K must be as in (5) and (6), and deleting at the outset terms not compatible with the prescribed boundary conditions at infinity, we have then that w and K will be of the form

$$w = - \frac{Pa^2 \sin 2\theta}{2(1-\nu)D} \left(\frac{1}{2} \frac{r^2}{a^2} + c_1 \frac{a^2}{r^2} + c_3 \ker_2 \lambda r + c_4 \kei_2 \lambda r \right) , \tag{16}$$

$$K = \frac{\frac{1}{2} Pa^2 \sin 2\theta}{(1-\nu) \sqrt{DB}} \left(c_2 \frac{a^2}{r^2} - c_3 \kei_2 \lambda r + c_4 \ker_2 \lambda r \right) , \tag{17}$$

with four arbitrary constants c_n, and with the Kelvin functions $\ker_2$ and

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kei_2 subject to the two ordinary second order differential equations

$$ker_2''x + x^{-1}ker_2'x - 4x^{-2}ker_2x = -kei_2x , \qquad (18a)$$

$$kei_2''x + x^{-1}kei_2'x - 4x^{-2}kei_2x = ker_2x . \qquad (18b)$$

In deriving expressions for stress resultants and couples from (16) and (17), it will be convenient to introduce the abbreviations

$$ker_2 = k_r , \quad kei_2 = k_i ; \quad \lambda r = x , \quad \lambda a = \mu . \qquad (19)$$

Therewith, and with (18a,b), we obtain from equations (1) and (3)

$$N_{rr} = \frac{\frac{1}{2} Pa \sin 2\theta}{(1-\nu)\sqrt{DB}} \left\{ -6c_2 \frac{a^4}{r^4} - \mu^2 \left[c_3 \left(\frac{k_i'}{x} - \frac{4k_i}{x^2} \right) - c_4 \left(\frac{k_r'}{x} - \frac{4k_r}{x^2} \right) \right] \right\} , \qquad (20)$$

$$N_{r\theta} = \frac{\frac{1}{2} P \cos 2\theta}{(1-\nu)\sqrt{DB}} \left\{ -6c_2 \frac{a^4}{r^4} - \mu^2 \left[2c_3 \left(\frac{k_i'}{x} - \frac{k_i}{x^2} \right) - 2c_4 \left(\frac{k_r'}{x} - \frac{k_r}{x^2} \right) \right] \right\} , \qquad (21)$$

$$M_{rr} = \frac{P \sin 2\theta}{2(1-\nu)} \left\{ -\mu^2 [c_3 k_i - c_4 k_r] - (1-\nu) \left(-1 - 6c_1 \frac{a^4}{r^4} \right. \right.$$

$$\left. \left. + \mu^2 \left[c_3 \left(\frac{k_r'}{x} - \frac{4k_r}{x^2} \right) + c_4 \left(\frac{k_i'}{x} - \frac{4k_i}{x^2} \right) \right] \right) \right\} , \qquad (22)$$

$$Q_r + \frac{M_{r\theta,\theta}}{r} = -\frac{P \sin 2\theta}{2(1-\nu)a} \left\{ \mu^3 [c_3 k_i' - c_4 k_r'] \right\}$$

$$- \frac{P \sin 2\theta}{r} \left\{ 1 - 6c_1 \frac{a^4}{r^4} + 2\mu^2 \left[c_3 \left(\frac{k_r'}{x} - \frac{k_r}{x^2} \right) + c_4 \left(\frac{k_i'}{x} - \frac{k_i}{x^2} \right) \right] \right\} . \qquad (23)$$

Introduction of (20) to (23) into the boundary conditions (14) then leads to the following set of four simultaneous equations for the determination of the four constants of integration c_n.

$$c_3 (\mu k_i' - 4k_i) - c_4 (\mu k_r' - 4k_r) = -6c_2 , \qquad (24)$$

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$$c_3(\mu k_i' - k_1) - c_4(\mu k_r' - k_r) = -3c_2 \quad , \tag{25}$$

$$c_3\left(\frac{\mu^2 k_1}{1-\nu} + \mu k_r' - 4k_r\right) - c_4\left(\frac{\mu^2 k_r}{1-\nu} - \mu k_i' + 4k_1\right) = 6c_1 + 1 \quad , \tag{26}$$

$$c_3\left(\frac{\mu^3 k_i'}{1-\nu} + 4\mu k_r' - 4k_r\right) - c_4\left(\frac{\mu^3 k_r'}{1-\nu} - 4\mu k_i' + 4k_1\right) = 12c_1 - 2 \quad , \tag{27}$$

where now $k_1 \equiv k_1(\mu)$, etc.

Upon suitable transformations, this system of equations can be written in a somewhat simpler form. To begin with, equations (24) and (25) are readily shown to be equivalent to the set[†]

$$-c_2 + c_3 k_1 - c_4 k_r = 0 \quad , \tag{24'}$$

$$2c_2 + c_3 \mu k_i' - c_4 \mu k_r' = 0 \quad . \tag{25'}$$

Having (24') and (25'), we may use (26) and (27) so as to obtain in place of these two equations the set

$$-2c_1 - \frac{\mu^2}{1-\nu} c_2 + c_3 \mu k_r' + c_4 \mu k_i' = -1 \quad , \tag{26'}$$

$$c_1 - \frac{1}{2}\frac{\mu^2}{1-\nu} c_2 + c_3 k_r + c_4 k_1 = -\frac{1}{2} \quad . \tag{27'}$$

Before evaluating the system (24') to (27'), it is useful to establish the analytical form of the quantities which are of principal physical interest. These quantities are the edge values of the couple $M_{\theta\theta}$ and of the resultant $N_{\theta\theta}$. We obtain a particularly convenient form of these expressions by making use of equations (1) and (2), in conjunction with

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[†] Corresponding to the fact that the conditions $N_{rr} = N_{r\theta} = 0$ for $r = a$ can be shown to be equivalent to conditions $K = K_{,r} = 0$.

two of the boundary conditions in (14), so as to have

$$M_{\theta\theta}(a,\theta) = -(1+\nu)D\nabla^2 w(a,\theta) , \qquad N_{\theta\theta}(a,\theta) = \nabla^2 K(a,\theta) \qquad (28)$$

An introduction of (16) and (17) into (28) gives, with the help of (18a,b),

$$M_{\theta\theta}(a,\theta) = -\frac{P}{2}\frac{1+\nu}{1-\nu}\mu^2(c_3 k_i - c_4 k_r)\sin 2\theta , \qquad (29a)$$

$$N_{\theta\theta}(a,\theta) = -\frac{P}{2}\frac{\mu^2}{(1-\nu)\sqrt{DB}}(c_3 k_r + c_4 k_i)\sin 2\theta . \qquad (29b)$$

Having (29a,b) we see, with the help of (24') and (27'), the possibility of the further relations

$$M_{\theta\theta}\left(a, \frac{\pi}{4}\right) = -\frac{P}{2}\frac{1+\nu}{1-\nu}\mu^2 c_2 , \qquad (30a)$$

$$N_{\theta\theta}\left(a, \frac{\pi}{4}\right) = \frac{P}{2}\frac{\mu^2}{(1-\nu)\sqrt{DB}}\left(\frac{1}{2} + c_1 - \frac{1}{2}\frac{\mu^2}{1-\nu}c_2\right) , \qquad (30b)$$

and it remains only to determine the coefficients c_1 and c_2 from equations (24') to (27'). We do this by first expressing c_3 and c_4 in terms of c_2, from (24') and (25'), in the form

$$c_3 = -\frac{c_2}{\mu}\frac{\mu k_r' + 2k_r}{k_i' k_r - k_r' k_i} , \qquad c_4 = -\frac{c_2}{\mu}\frac{\mu k_i' + 2k_i}{k_i' k_r - k_r' k_i} , \qquad (31)$$

and by then using (26') and (27') in order to obtain the relations

$$c_2 = \frac{1-\nu}{\mu^2}\left[1 + (1-\nu)\frac{(\mu k_r' + 2k_r)^2 + (\mu k_i' + 2k_i)^2}{2\mu^3(k_i' k_r - k_r' k_i)}\right]^{-1} , \qquad (32a)$$

$$\frac{1}{2} + c_1 - \frac{c_2}{2}\frac{\mu^2}{1-\nu} = \frac{k_r(\mu k_r' + 2k_r) + k_i(\mu k_i' + 2k_i)}{\mu(k_i' k_r - k_r' k_i)}c_2 . \qquad (32b)$$

It is possible to simplify the form of (32a,b) somewhat by making use of certain identities involving Kelvin functions of various orders.

In this way we obtain[†], upon introducing (32a,b) into (30a,b), as expressions for the significant edge moment and the significant edge resultant, in terms of $\underline{\text{zeroth}}$ order Kelvin functions,

$$\frac{M_{\theta\theta}(a,\pi/4)}{-P/2} = \frac{1+\nu}{1+(1-\nu)f_1} \tag{33a}$$

where

$$f_1 = \frac{1}{2\mu} \frac{(\text{ker}'\,\mu)^2 + (\text{kei}'\,\mu)^2}{\text{kei}'\,\mu\;\text{ker}\,\mu - \text{ker}'\,\mu\;\text{kei}\,\mu - 2\mu^{-1}[(\text{ker}'\,\mu)^2 + (\text{kei}'\,\mu)^2]} \tag{33b}$$

and

$$\frac{N_{\theta\theta}(a,\pi/4)}{P/2} = \frac{1}{\sqrt{DB}} \frac{f_2}{1+(1-\nu)f_1} \tag{34a}$$

where

$$f_2 = \frac{\text{kei}'\,\mu\;\text{kei}\,\mu + \text{ker}'\,\mu\;\text{ker}\,\mu}{\text{kei}'\,\mu\;\text{ker}\,\mu - \text{ker}'\,\mu\;\text{kei}\,\mu - 2\mu^{-1}[(\text{ker}'\,\mu)^2 + (\text{kei}'\,\mu)^2]} \tag{34b}$$

Stress concentration factors for bending stresses and membrane stresses. We define a bending stress concentration factor k_b as the ratio $M_{\theta\theta}(a,\pi/4)/M_o$ where $M_o = M_{\theta\theta}(\infty,\pi/4) = -P/2$. Therewith $\underline{k_b\text{ is}}$ directly given by the right-hand side in (33a).

In order to obtain the corresponding membrane stress concentration factor k_m, it is necessary to be more specific about the nature of the two-dimensionally isotropic shell medium. We shall assume in what follows that the shell is homogeneous in thickness direction and have then the relation

$$DB = \frac{Eh^3}{12(1-\nu^2)} \frac{1}{Eh} = \frac{h^2}{12(1-\nu^2)} \tag{35a}$$

[†] See equations (9.9.14) to (9.9.17) in [1].

We write further

$$\sigma_m = \frac{N_{\theta\theta}(a, \pi/4)}{h} , \qquad \sigma_o = \frac{6M_o}{h^2} = -\frac{3P}{h^2} , \tag{35b}$$

and therewith obtain from (34a)

$$k_m = \frac{\sigma_m}{\sigma_o} = \sqrt{\frac{1-\nu^2}{3}} \frac{f_2}{1 + (1-\nu)f_1} . \tag{35c}$$

$\underline{\text{Stress concentration factors for small and for large values of } \mu.}$
Given that $\mu = \lambda a = a/\sqrt[4]{R^2 BD} = \sqrt[4]{12(1-\nu^2)} a/\sqrt{Rh}$, the limiting case of a
flat plate corresponds to the assumption $\mu = 0$. We find, from equations
(33b) and (34b), that $f_1(0) = -1/4$ and $f_2(0) = 0$ and therewith from
(33a) and (35c),

$$(k_b)_{\mu=0} = \frac{4 + 4\nu}{3 + \nu} , \qquad (k_m)_{\mu=0} = 0 , \tag{36a,b}$$

with this result coinciding, as it should, with Goodier's result
for plates, without consideration of transverse shear deformation [2].

For the case of large μ, corresponding to a shell problem with
distinct interior and edge zone solution contributions use may be made
of appropriate asymototic formulas. We find, by making use of certain
known cross-product expansion formulas[†] that

$$f_1 \approx -\frac{\sqrt{2}}{2\mu} , \qquad f_2 \approx 1 - \frac{3\sqrt{2}}{2\mu} + \ldots \tag{37a,b}$$

and therewith,

[†] Equations (9.10.32) to (9.10.34) in [1].

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$$k_b \approx (1+\nu)\left(1 + \frac{1-\nu}{\mu\sqrt{2}}\right) \quad , \quad k_m \approx \sqrt{\frac{1-\nu^2}{3}}\left(1 - \frac{2+\nu}{\mu\sqrt{2}}\right) \qquad (38a,b)$$

except for terms of relative order $1/\mu^2$.

Inasmuch as bending and membrane stresses superimpose the relevant stress concentration factor for the most highly stressed face of the shell comes out to be, for sufficiently large values of μ,

$$k = k_b + k_m \approx 1 + \nu + \sqrt{\frac{1-\nu^2}{3}} - \frac{\sqrt{1-\nu^2}}{\mu\sqrt{2}}\left(\frac{2+\nu}{\sqrt{3}} - \sqrt{1-\nu^2}\right) . \qquad (39)$$

It may be noted that the numerical values of k for $\mu = 0$ and for $\mu = \infty$ are not greatly different, but that while for $\mu = 0$ the stress concentration is due entirely to bending, a significant fraction of it is, for $1 \ll \mu$, due to membrane rather than due to bending action. Numerical values for f_1, f_2, k_b and k_m, as a function of μ and ν, may be found in Table 1.

Interior solution stresses for large μ. The form of the expressions (16) and (17) for w and k indicates that for large values of μ the effect of the terms with c_3 and c_4 is significant in a narrow edge zone only and that outside this zone the remaining expression for w is as if bending occurred without stretching and the remaining expression for K is as if the state of stress of the shell was a pure membrane state.

We obtain information on the state of stress outside the narrow edge zone, and in particular on the relative significance of bending and membrane stresses, by determining the values of $M_{\theta\theta}$ and $N_{\theta\theta}$ in accordance with (16), (17) and the defining relations (1) and (2), by setting $c_3 = c_4 = 0$ in (16) and (17) and by then deriving the relations

$$M_{\theta\theta}^{(i)}\left(a, \frac{\pi}{4}\right) \approx -\frac{P}{2}(1 + 6c_1) \quad , \quad N_{\theta\theta}^{(i)}\left(a, \frac{\pi}{4}\right) \approx \frac{P}{2}\frac{6c_2}{(1-\nu)\sqrt{DB}} \qquad (40a,b)$$

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We evaluate (40a) by taking c_1 from equation (26), with c_3 and c_4 as in (31) and (32a). Therewith we obtain, except for terms small of higher order

$$\frac{M_{\theta\theta}^{(i)}}{M_0} = \frac{\sigma_b^{(i)}}{\sigma_0} \approx c_2 \frac{\mu_2}{1-\nu} \approx 1 \tag{41a}$$

A corresponding evaluation of (40b) leads to the relations

$$\frac{hN_{\theta\theta}^{(i)}}{6M_0} = \frac{\sigma_m^{(i)}}{\sigma_0} \approx -c_2 \sqrt{12 \frac{1+\nu}{1-\nu}} \approx -\frac{\sqrt{12(1-\nu^2)}}{\mu^2} \tag{41b}$$

A comparison of (41a,b) with (38a,b) shows that the order of magnitude of the bending stress in the interior is the same as the order of magnitude of this stress in the edge zone, in such a way that the dimensionless edge zone value $1+\nu$ decreases to a value 1 in the interior. At the same time the interior membrane stress comes out to be small of relative order $1/\mu^2$ so that, effectively, the interior state of the shell is a state of inextensional bending.

Direct Asymptotic Solution for Interior and Edge Zone States. We proceed as in [4] to solve the given boundary value problem, for values of μ which are sufficiently large compared to unity, through use of equations (5) to (13). Introduction of (7) and (13) into the two sets of boundary conditions (14) and (15) then leaves as conditions for the determination of the two harmonic functions ϕ and ψ and of the "plate on an elastic foundation" function χ, for $r = a$,

$$\psi_{,rr} + RD\left(\frac{(\nabla^2\chi)_{,r}}{r} + \frac{(\nabla^2\chi)_{,\theta\theta}}{r^2}\right) = 0 . \tag{42}$$

$$\frac{\psi_{,r\theta}}{r} - \frac{\psi_{,\theta}}{r^2} - RD\left(\frac{(\nabla^2\chi)_{,r\theta}}{r} - \frac{(\nabla^2\chi)_{,\theta}}{r^2}\right) = 0 . \tag{43}$$

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$$(1-\nu)\phi_{,rr} + \nabla^2\chi + (1-\nu)\left(\frac{\chi_{,r}}{r} + \frac{\chi_{,\theta\theta}}{r^2}\right) = 0 \quad , \tag{44}$$

$$\frac{1-\nu}{r}\left(\frac{\phi_{,r\theta}}{r} - \frac{\phi_{,\theta}}{r^2}\right)_{,\theta} + \frac{1-\nu}{r}\left(\frac{\chi_{,r\theta}}{r} - \frac{\chi_{,\theta}}{r}\right)_{,\theta} + (\nabla^2\chi)_{,r} = 0 \quad , \tag{45}$$

with equations (42), (43) and (45) also holding for $r = \infty$, and with the right-hand side of (44) being replaced by $-(P/2D)\sin 2\theta$ for $r = \infty$.

We now note that when $1 \ll \mu$ we have the order of magnitude relations,

$$\chi = O(a\chi_{,r}) \quad , \quad \chi_{,r} = O(a\chi_{,rr}) \quad , \tag{46}$$

etc. We use these for an asymptotic solution of the problem, by retaining in (44) and (45) the highest and second highest order of magnitude terms in χ, $(\nabla^2\chi)_{,r}$ and $\nabla^2\chi$, only, that is, we replace equations (44) and (45) by the abbreviated equations

$$(1-\nu)\phi_{,rr} + \nabla^2\chi = 0 \quad , \quad \frac{1-\nu}{r}\left(\frac{\phi_{,r}}{r} - \frac{\phi}{r^2}\right)_{,\theta\theta} + (\nabla^2\chi)_{,r} = 0 \quad . \tag{47,48}$$

An introduction of this into (42) and (43) then leaves as two conditions for the determination of the two harmonic functions ϕ and ψ.[†]

$$\psi_{,rr} - RD\frac{1-\nu}{r^2}\left|\left(\frac{\phi_{,r}}{r} - \frac{\phi}{r^2}\right) + \phi_{,rr}\right|_{,\theta\theta} = 0 \quad , \tag{49}$$

$$\frac{\psi_{,r\theta}}{r} - \frac{\psi_{,\theta}}{r^2} + RD\frac{1-\nu}{r^2}\left|\left(\frac{\phi_{,r}}{r} - \frac{\phi}{r^2}\right)_{,\theta\theta} - \phi_{,rr}\right|_{,\theta} = 0 \quad , \tag{50}$$

[†]Note that upon writing equations (7) to (13) in the form $N_r = N_r^i + N_r^e$, etc., so as to distinguish between interior and edge zone solution contributions, equations (49) and (50) are equivalent to the previously derived contracted boundary conditions for the determination of the interior state [4], of the form $r^2 N_{rr}^i - R(M_{r\theta}^i + M_{rr,\theta}^i)_{,\theta} = r^2 N_{r\theta}^i - R(M_{r\theta,r}^i - M_{rr}^i)_{,\theta} = 0$.

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Having determined ϕ and ψ, we subsequently determine the associated approximation for the edge zone function χ with the help of equations (48)[†], and we use the results obtained in this way in order to obtain from equations (8) and (12) as approximate expressions for the relevant edge values of circumferential stress resultant and stress couple

$$N_{\theta\theta} = \psi_{,rr} + RD\lambda^4\chi \quad , \quad M_{\theta\theta} = -(1-\nu)D\phi_{,rr} - \nu D\nabla^2\chi \quad , \tag{51}$$

for $r = a$.

In order to carry out the remaining simple calculations we write, consistent with (16) and (17), in order to assure satisfaction of all conditions at infinity

$$\phi = w^1 = -\frac{Pa^2 \sin 2\theta}{2(1-\nu)D}\left(\frac{1}{2}\frac{r^2}{a^2} + c_1\frac{a^2}{r^2}\right) \quad , \tag{52}$$

$$\psi = K^1 = \frac{Pa^2 \sin 2\theta}{2(1-\nu)\sqrt{DB}}\left(c_2\frac{a^2}{r^2}\right) \quad , \tag{53}$$

and we further write

$$\chi = e^{-\lambda(r-a)/\sqrt{2}}\left(C_3 \cos \lambda\frac{r-a}{\sqrt{2}} + C_4 \sin \lambda\frac{r-a}{\sqrt{2}}\right) \quad , \tag{54}$$

and

$$\nabla^2\chi = \lambda^2 e^{-\lambda(r-a)/\sqrt{2}}\left(C_3 \sin \lambda\frac{r-a}{\sqrt{2}} - C_4 \cos \lambda\frac{r-a}{\sqrt{2}}\right) \quad . \tag{55}$$

We now introduce (52) and (53) into the boundary conditions (49) and (50) and obtain as two equations for the determination of c_1 and c_2

$$c_2\mu^2 - (1-\nu)(1+6c_1) = 0 \quad , \quad c_2\mu^2 - (1-\nu)(1-6c_1) = 0 \quad . \tag{56}$$

[†] We note that these equations may be written, equivalently, as
$D\nabla^2\chi = M_{rr}^1$ and $D(\nabla^2\chi)_{,r} = r^{-1}M_{r\theta,\theta}^1 = 0$, for $r = a$.

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Equations (56) imply, consistent with (32), that

$$c_2\mu^2 = 1 - \nu \ , \quad c_1 = 0 \quad . \tag{57}$$

Having c_2 and c_1 as in (57), we finally obtain C_3 and C_4 from (48) in the form

$$C_3 = -P/2D\lambda^2 \ , \quad C_4 = -C_3 - P\sqrt{2}/D\lambda^3 a \approx -C_3 \ , \tag{58}$$

and therewith, from (51),

$$N_{\theta\theta}\left(a, \frac{\pi}{4}\right) = -\frac{P}{2\sqrt{DB}} \ , \quad M_{\theta\theta}\left(a, \frac{\pi}{4}\right) = -(1+\nu)\frac{P}{2} \ . \tag{59}$$

The above expressions for the edge values of $N_{\theta\theta}$ and $M_{\theta\theta}$ may be compared with the interior values of these same two quantities, $N^i_{\theta\theta}(a,\pi/4)$ = $6P/2\sqrt{DB}\,\mu^2$ and $M^i_{\theta\theta}(a,\pi/4) = -P/2$, which follow from (52),(53) and (57), consistent with the contents of equation (41).

Table 1

μ	f_1	f_2	k_b			k_m		
			$\nu=0$	$\nu=1/3$	$\nu=1/2$	$\nu=0$	$\nu=1/3$	$\nu=1/2$
0	-0.250	0	1.333	1.600	1.714	0	0	0
.1	-0.249	0.012	1.332	1.599	1.713	0.009	0.008	0.007
.3	-0.243	0.063	1.321	1.591	1.707	0.048	0.041	0.036
.5	-0.234	0.122	1.305	1.580	1.699	0.092	0.078	0.069
.8	-0.219	0.206	1.280	1.561	1.684	0.152	0.131	0.116
1	-0.208	0.257	1.263	1.548	1.674	0.187	0.162	0.143
2	-0.165	0.443	1.198	1.498	1.635	0.306	0.271	0.241
3	-0.135	0.557	1.56	1.465	1.609	0.372	0.333	0.299
4	-0.114	0.633	1.128	1.443	1.591	0.413	0.373	0.336
5	-0.098	0.687	1.109	1.427	1.578	0.440	0.400	0.362
∞	0	1.000	1.000	1.333	1.500	0.577	0.544	0.500

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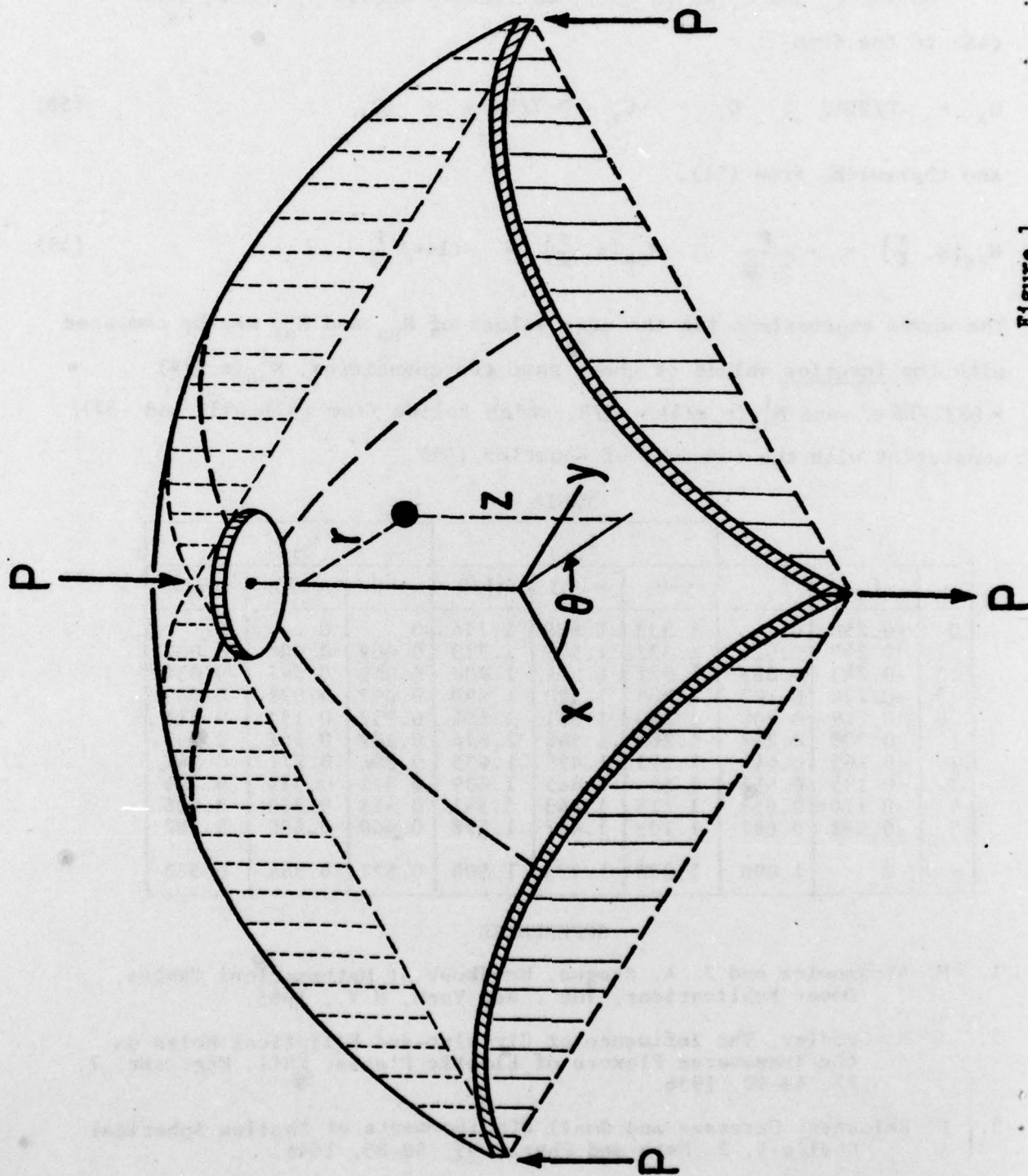
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Figure 1

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Shallow spherical shell, transverse twisting, small circular hole, stress concentration, closed-form solution, edge-zone behavior, asymptotic determination, interior and edge-zone contributions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The problem of transverse twisting of a shallow spherical shell with a small circular hole is solved, in generalization of the corresponding problem of a flat plate. The solution is of interest as a closed-form solution of an unsymmetrical stress concentration problem, with quantitative features depending on its boundary layer behavior for large values of a relevant parameter. The problem is also of interest as an example of the applicability of a previously proposed asymptotic procedure where interior contributions and edge-zone contributions are determined in sequence rather than simultaneously.